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Numerical Solution of Magnetohydrodynamic Stagnation-Point Flow Equations

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In a problem on the magnetohydrodynamic, axisymmetric flow of a conducting fluid near a stagnation point recently considered by the writer,[3] similarity solutions of the Navier-Stokes and Maxwell equations are obtained from the solution of the ordinary differential equations

$$(F'''/2) + (FF''/2) - \tfrac{1}{4}(F'^2 - 1) = \beta(H^2 - 1) \quad (1)$$

$$H'' + \epsilon FH' = 0 \quad (2)$$

satisfying $F(0) = F'(0) = H(0) = 0$; $F'(\infty) = H(\infty) = 1$. The stream function and magnetic field are effectively F and H, and β and ϵ are parameters defined in Ref. 3.

In the present note we describe a method of solving this coupled system of equations by successive approximations based on a method given by Weyl[5] for the nonmagnetic case. Weyl replaced the equation corresponding to (1) for F by an integral equation for F'', then approximated F'' in the integrand. We extend this idea to include (2) and choose for the initial approximation an especially simple form for F. Recently, Davies[1,2] has applied a similar but rather more sophisticated process on a similar pair of equations.

Instead of (1) and (2) we consider the equivalent system,

$$F^{iv} + FF''' - 4\beta HH' \quad (3)$$

$$H'' + \epsilon FH' = 0 \quad (4)$$

satisfying

$$F(0) = F'(0) = H(0) = 0$$

$$F'''(0) = -\tfrac{1}{2} - 2\beta \qquad F'(\infty) = H(\infty) = 1$$

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For F known, (4) can be solved to yield

$$H(\zeta) = b\int_0^\zeta \exp\left[-\epsilon\int_0^s F(t)dt\right]ds$$

where

$$\frac{1}{b} = \int_0^\infty \exp\left[-\epsilon\int_0^s F(t)dt\right]ds$$

Following Weyl, we integrate (3) twice to produce the integral equation

$$G(\zeta) = \left(\frac{1}{2} + 2\beta\right)\int_\zeta^\infty \exp\left[-\int_0^s \frac{(s-t)^2G(t)dt}{2}\right]ds - 4\beta b^2\int_\zeta^\infty \exp\left[-\int_0^s \frac{(s-t)^2G(t)dt}{2}\right]\times \int_0^s \exp\left[(1-\epsilon)\int_0^t \frac{(t-u)^2G(u)du}{2}\right]\times \int_0^t \exp\left[-\epsilon\int_0^u \frac{(u-v)^2G(v)dv}{2}\right]dt\,ds \quad (5)$$

for $F''(\zeta) = G(\zeta)$. The case $\beta = 0$ gives the integral equation discussed by Weyl apart from a trivial modification in F and ζ. For a given approximation G_n, (5) determines the next approximation G_{n+1}. However, here we choose to formulate the successive approximation scheme by means of the equations

$$F_{n+1}{}^{iv} + F_nF'''_{n+1} = 4\beta H_nH_n' \quad (6)$$

$$H_n'' + \epsilon F_nH_n' = 0 \quad (7)$$

satisfying

$$F_{n+1}(0) = F'_{n+1}(0) = H_n(0) = 0$$

$$F'''_{n+1}(0) = -\tfrac{1}{2} - 2\beta$$

$$F'_{n+1}(\infty) = H_n(\infty) = 1$$

This is equivalent to the Weyl scheme but allows us to use, for our initial approximation, the function $F_1 = c\zeta$, ζ being the independent variable. Equation (7) then yields H_1, (6) yields F_2, and so on. The constant c is chosen so that all the boundary conditions are satisfied by H_1 and F_2, and in addition $F_2''(\infty) = 0$. We find

$$H_1(\zeta) = (2\epsilon c)^{1/2}\int_0^\zeta \exp\left(\frac{-\epsilon cs^2}{2}\right)ds \Big/ \left(-\frac{1}{2}\right)!$$

and, on integrating (6) three times (see Appendix),

$$F_2'(\zeta) = \int_0^\zeta\int_\infty^v \exp\left(\frac{-cu^2}{2}\right)\left\{\frac{8\beta\epsilon c}{[(-\frac{1}{2})!]^2}\times \int_0^u \exp\left[\frac{(1-\epsilon)ct^2}{2}\right]\int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right]ds\,dt - \left(\frac{1}{2} + 2\beta\right)\right\}du\,dv \quad (8)$$

The boundary conditions $H_1(0) = 0$, $H_1(\infty) = 1$, $F_2'''(0) = -\frac{1}{2} - 2\beta$, $F_2''(\infty) = 0$, and $F_2'(0) = 0$ are all satisfied, and c is now determined by the equation $F_2'(\infty) = 1$, after which a further integration yields $F_2(\zeta)$, on applying the condition $F_2(0) = 0$. After integrating (8) by parts and some manipulation we find $c = \frac{1}{2}$. Note that c is independent of β and ϵ so that H_1 is known once and for all as an error function, $H_1 = \text{erf}(\epsilon^{1/2}\zeta/2)/(\frac{1}{2})!$, where

$$\text{erf}x = \int_0^x e^{-u^2}du$$

The writer had two exact solutions of (1) and (2), corresponding to $\beta = 0$, $H = 0$, the well-known Homann solution (see, e.g., Ref. 4), and $\beta = 1$, $\epsilon = \frac{1}{16}$ (see Ref. 3). In the former case, it is found that F_2 and its first two derivatives contain errors of order, at most, 6% over the range of ζ of practical interest. Up to $\zeta = 1.4$, $F_2''(\zeta)$ underestimates the value (by less than 5% at $\zeta = 0$) and then overestimates, whereas F_2' and F_2 underestimate for all ζ.

In the latter case, H_1 and F_2 have errors of the order of 30% which get much bigger for certain ranges of ζ. The functions H_2 and F_3 were calculated for this case, and they led to a noticeable improvement in accuracy, the errors then in H_2 and H_2' being roughly 20%, whereas those in the F_3 functions were 10–15%. More important, perhaps, bearing in mind the "pincer-movement" detected by Weyl for the $\beta = 0$ case when establishing convergence, we find the same phenomenon appearing to begin here, viz., $F_2 > F > F_3$ and $H_1 < H < H_2$ for all ζ.

Finally, in connection with the possible use of the method in predicting rough starting values for use in standard step-by-step methods of numerical integration we note that the error in $H_1'(0)$ is 39% and in $H_2'(0)$ is 26%, whereas in $F_2''(0)$ it is 32% and in $F_3''(0)$, 12.%

Appendix

The derivation of formula (8) for $F_2'(\zeta)$ requires the vanishing of the following expressions:

$$\left[\zeta \int_{\infty}^{\zeta} \exp\left[\frac{-cu^2}{2}\right] \int_0^u \exp\left[\frac{(1-\epsilon)ct^2}{2}\right] \times \int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds\, dt\, du\right]_0^{\infty} \quad (9)$$

$$\left\{\exp\left[\frac{-cv^2}{2}\right] \int_0^v \exp\left[\frac{(1-\epsilon)ct^2}{2}\right] \times \int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds\, dt\right\}_0^{\infty} \quad (10)$$

In proving that these expressions are zero, we make use of the inequality,

$$\int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds \leqslant t$$

$$0 \leqslant t < \infty \qquad (\epsilon, c \geqslant 0) \quad (11)$$

Then the magnitude of the bracketed expression in (9) is

$$\leqslant \frac{\zeta}{c|1-\epsilon|} \int_{\zeta}^{\infty} \left| \exp\left[\frac{-\epsilon cu^2}{2}\right] - \exp\left[\frac{-cu^2}{2}\right] \right| du \quad (12)$$

It follows that (9) vanishes at the upper limit because

$$\zeta \int_{\zeta}^{\infty} e^{-u^2}\, du \leqslant \zeta \int_{\zeta}^{\infty} e^{-u} du = \zeta e^{-\zeta} \to 0 \text{ as } \zeta \to \infty$$

and at the lower limit because both terms in the integral in (12) can be transformed into error functions and have finite constant values at $\zeta = 0$.

Similarly, by using (11), we note that the magnitude of the bracketed expression in (10) is

$$\leqslant \frac{\exp(-\epsilon cv^2/2) - \exp(-cv^2/2)}{c(1-\epsilon)}$$

which clearly tends to zero as $v \to \infty$ for all positive ϵ. At the lower limit we observe that the integral in (10) $\sim v^2/2$ for small v which establishes the result.

The preceding argument is not valid as it stands when $\epsilon = 1$, but in that case the original expressions (9) and (10) are simplified, and minor modifications may be introduced to show that the results are still true.

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Effect of Thermal Radiation on Thin Shock Structure

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1. Introduction

In this note we shall report some findings through the application of "boundary-layer technique" on the problem of shock transition with thermal radiation. We approach the problem of studying shock transition by considering the viscosity and the heat-conduction effects of the gas in addition to the radiation effect. Then, in the limiting process, we examine the physical situation corresponding to a shock of vanishing thickness. Whenever we refer to the shock in the following we mean only the region in which viscosity and heat conduction are important. The mathematical theory of the study of the nonradiating shock transition has been very well established.[3] The boundary-layer nature of the shock transition allows one to study the flow field of the shock structure as well as the flow field outside the shock in a systematic manner based on a small parameter $1/Re$, where Re is the reference Reynolds number.[4] This technique has been used in the present analysis of the study of the shock transition with thermal-radiation effect. It will be seen that in this case, a new parameter $\nu = 0(\bar{\alpha}/Re)$ will play an important role in the analysis, where $\bar{\alpha}$ is the reference value of the absorption coefficient of the gas medium. In the limit as ν approaches zero, the flow field will correspond to the case of an infinitesimally thin shock. It will also be seen that the shock structure of the lowest order is modified due to the presence of the thermal-radiation effect. Included in Ref. 5 and 8 are some altogether differenta pproaches to the radiative shock-structure problem.

2. Governing Flow Equations

The basic equations for the steady one-dimensional flow can be expressed in the following form:

$$d/dx\ (\rho u) = 0 \quad (1a)$$

$$(\rho u)\ du/dx + dp/dx - d\tau/dx = 0 \quad (1b)$$

$$(\rho u)\ d/dx(h + \tfrac{1}{2}u^2) + d/dx(q + q_r - u\tau) = 0 \quad (1c)$$

$$p = [(\gamma - 1)/\gamma]\,\rho h \quad \text{(perfect gas)}$$

These equations can be integrated once to give

$$\rho u = \Gamma \quad (2a)$$

$$\Gamma u + [(\gamma-1)/\gamma]\Gamma h/u - \tau = \Gamma c_1 \quad (2b)$$

$$\Gamma(h + \tfrac{1}{2}u^2) + (q + q_r - u\tau) = \Gamma c_2 \quad (2c)$$

The shearing stress τ and the heat flux due to heat conduction q can be expressed as $\tau = \mu/L\ (du/dx)$ and $q = -k/Lc_p\ (dh/$

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